Time-Dependent Solutions of Einstein's Equations

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Spherically symmetric time-dependent solutions are presented. One class of solutions could represent the interior of an incompressible sphere undergoing at its surface a process of condensation or evaporation. A large class of solutions of the equation $T_2^2=0$ is also obtained. A generalization of the Oppenheimer-Snyder solution is found. Two solutions obeying an equation of state are described.

We shall suppose that the reader is aware of the importance of time-dependent solutions.[1] We shall therefore not elaborate on this subject and shall present directly our results.

I. CONDENSATION AND EVAPORATION AT THE SURFACE OF AN INCOMPRESSIBLE BODY

Let us consider the element

$$ds^2=-A(r)dr^2-r^2(d\theta^2+\sin^2\theta d\psi^2)+C(r)dt^2, \qquad (1)$$

which will be written alternatively as

$$ds^2=-e^{\omega}dr^2-r^2d\Omega^2+e^{\sigma}dt^2. \qquad (2)$$

The equation $T_1^1=T_2^2$ reduces to[2]

$$-\frac{A'}{A^2}\left(\frac{C'}{4C^2}+\frac{1}{2r}\right)+\frac{1}{A}\left(\frac{C''}{2C}-\frac{C'}{2rC}-\frac{1}{r^2}\right)+\frac{1}{r^2}=0. \qquad (3)$$

It can also be written

$$2\sigma''+\sigma'^2-\sigma'\left(\frac{2}{r}+\frac{A'}{A}\right)=\frac{-2A'}{2A}+\frac{4}{r^2}-\frac{4A}{r^2}. \qquad (4)$$

This equation was solved giving A in terms of an arbitrary function σ. However, if we consider Eq. (4) as a differential in σ (A being an arbitrary function), it becomes a Riccati differential equation that can be solved only if we know a particular solution σ_0. Taking the particular solution as a parameter, we can solve Eq. (4) and obtain

$$C=e^{\sigma}=e^{\sigma_0}\left(h\int r\sqrt{A}\,e^{-\sigma_0}dr+g\right)^2 \qquad (5)$$

(h and g are arbitrary constants). A is given by[2]

$$A=\frac{e^{\sigma_0}(2+r\sigma_0')^2\exp\left(-4\int\frac{\sigma_0'dr}{2+r\sigma_0'}\right)}{-4r^2\int\left[\frac{e^{\sigma_0}}{r^3}(2+r\sigma_0')\exp\left(-4\int\frac{\sigma_0'dr}{2+r\sigma_0'}\right)\right]dr+\text{const}\times r^2}. \qquad (6)$$

The interesting thing is that Einstein's time-dependent field equations remain satisfied if we consider h and g as arbitrary functions of time instead of being constants. The relevant equations are[3]

$$-8\pi T_1^1=e^{-\omega}\left(\frac{\sigma'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2}, \qquad (7)$$

$$-8\pi T_2^2=e^{-\omega}\left(\frac{\sigma''}{2}-\frac{\omega'\sigma'}{4}+\frac{\sigma'^2}{4}+\frac{\sigma'-\omega'}{2r}\right)$$

bitrary constant occurring in C *and not occurring in A* is replaced by an arbitrary function of time. Therefore, the equation

$$C=e^{\sigma}=e^{\sigma_0}\left[h(t)\int r\sqrt{A}\,e^{-\sigma_0}dr+g(t)\right]^2, \qquad (5')$$

together with Eq. (6), represents a comoving time-dependent solution for a perfect fluid.

More specifically, every static solution for $A=e^{\omega}$ and e^{σ_0} of Eqs. (7)–(10) generates a family of time-in which $r = R(t)$ is the equation of motion of the junction surface. The junction surface must therefore be comoving and, on it, the pressure of the interior solution must be zero. It should there-

undergoing a process of evaporation.

It may be objected that, in the case of condensation, the condensed matter would have different densities according to the coordinate radius at

[redacted]

for which the pressure is zero at all times. However, we would have on one hand from (7)

$$\sigma' = \frac{e^{\omega(R)}}{R} - \frac{1}{R},$$

which implies that, for the same value R of r, σ' is time-independent. However, we have from (5′)

$$\sigma' = \frac{\sigma_0' + 2h(t) r\sqrt{A}\, e^{-\sigma_0}}{h(t)\int r\sqrt{A}\, e^{-\sigma_0} dr + g(t)}.$$

This last expression can be time-independent only if $h(t)/g(t)$ is a constant. In this case $C = e^{\sigma}$ reduces

be held in case the interior solution is of constant density.

Such will be the case if we start with the Einstein static solution with $A = (1 - r^2/a^2)^{-1}$ and $e^{\sigma_0} = 1$. We obtain through Eq. (5′)

$$C = e^{\sigma} = [h(t)(1 - r^2/a^2)^{1/2} + g(t)]^2.$$

This solution is the Schwarzschild interior solution with two of the arbitrary constants replaced by functions of time. For such a solution the condensation at any coordinate radius will always give us matter with the same constant density.

[redacted]

This is a nonlinear differential equation that seems to be quite intractable; we have been able,

and a function of r. The first term gives the time-dependent mass singularity already computed,

$$A = B'^2/4B\,,$$
$$B = \dot{\tau}(t)^{-2/3}[F(r)\tau(t)+G(r)]^{4/3},\quad C=1, \tag{20}$$

in which F, G, and τ are arbitrary functions of their argument. We have in this case

$$T^1_4=0,\quad -8\pi(T^1_1=T^2_2=T^3_3) = -(\ddot{\tau}/\dot{\tau})^2+\tfrac{2}{3}\dddot{\tau}/\dot{\tau}. \tag{21}$$

The pressure is uniform through space, though time-dependent. Expression (21) for the pressure is invariant for any homographic transformation of τ with constant coefficients

$$\overline{\tau} = \frac{a\tau+b}{c\tau+d}\,. \tag{22}$$

For $\tau = t$ the solution (20) reduces to the OS solution; therefore, the OS solution can be obtained for the more general choice of

$$\tau = \frac{at+b}{cf+d}\,, \tag{23}$$

where a, b, c, and d are constants. The calculated density for the solution (20) is

take for G/F a continuous function which is constant over a number of nonoverlapping domains of r, we will have joined several Friedmannian regions in the most easy way. We may take for definiteness $F(r)=r^{3/2}$ (which fixes the meaning of the radial coordinate) and write $P=G/F=Gr^{-3/2}$ so that

$$B = r^2\dot{\tau}^{-2/3}[\tau(t)+P(r)]^{4/3},\quad A = B'^2/4B\,, \tag{28}$$

where

P is a constant P_1 for $r_1<r<r_2$,

P is a function of r for $r_2<r<r_3$, (29)

P is a constant P_2 for $r_3<r<r_4$.

The continuous function P in the interval $r_2<r<r_3$ is to be such that[8]

$$P(r_2)=P_1,\quad P(r_3)=P_2,\quad P'(r_2)=P'(r_3)=0\,. \tag{30}$$

The metric defined by (28) and (29) describes two Friedmannian regions (the first and last regions) a prerequisite in comoving coordinates; see Appendix A). The second solution corresponds to the following line element:

$$ds^2 = \frac{-2a}{1-r^2u}\,dr^2 - r^2\{a + C_1\sinh[(2c/a)^{1/2}t] + C_2\cosh[(2c/a)^{1/2}t]\}\,d\Omega^2 + crdt^2\,, \tag{34}$$

from which we calculate

$$\begin{aligned} &T^4_1 = 0\,,\\ &8\pi p = \frac{1}{r^2}\left(\frac{a^2 + C_2{}^2 - C_1{}^2}{2a\{a + C_1\sinh[(2c/a)^{1/2}t] + C_2\cosh[(2c/a)^{1/2}t]\}^2}\right) - \frac{3u}{2}\,,\\ &8\pi\rho = \frac{1}{r^2}\left(\frac{a^2 + C_2{}^2 - C_1{}^2}{2a\{a + C_1\sinh[(2c/a)^{1/2}t] + C_2\cosh[(2c/a)^{1/2}t]\}^2}\right) + \frac{3u}{2}\,, \end{aligned} \tag{35}$$

so that

$$\rho = p + 3u/8\pi\,. \tag{36}$$

For $C_1 = C_2 = 0$ the solution is static and can be smoothly joined to Schwarzschild's exterior solution with a coordinate radius

$$R = (3u)^{-1/2}$$

and a total mass

$$m = \tfrac{1}{3}(3u)^{-1/2}\,. \tag{37}$$

For $C_2 = -C_1$ the solution is time-dependent and tends asymptotically to the static solution as t tends to infinity. For $C_2 = C_1$ the solution is time-dependent and starts asymptotically from the static solution.

In order to avoid a singularity in the metric, as well as in the density and in the pressure at a finite time, we must have

$$C_2 > |C_1|\,. \tag{38}$$

The time-dependent solutions do not oscillate about the equilibrium position. However, it cannot be deduced from this that the static solution is unstable; in fact, none of the time-dependent solutions can be imbedded in a vacuum ($p = 0$ has no time-dependent solution for r; see Appendix A). The stability of the static solution must be judged from the behavior of perturbations compatible with the embedding in vacuum.

For $a^2 + C_2{}^2 - C_1{}^2 = 0$ we have a solution characterized by homogeneous pressure and density and by the relation $\rho + p = 0$. The solution, however, is different from De Sitter's solution, which has one more characteristic, namely, that of being isotropic at every point in comoving coordinates.

$r_2 > r_1 > u^{-1/2}$, we then have a shell of matter oscillating under the action of forces applied at the boundary surfaces $r = r_1$ and $r = r_2$. The value of the forces may be deduced from the expression of p for the corresponding values of r.

The two solutions of this paragraph have with Friedmann's solution the following characteristic: They are the only solutions representing a perfect fluid which, in comoving coordinates, correspond to expressions for A, B, and C which are products of a function of r by a function of t. Friedmann's solution has one more characteristic, namely, $dC/dr = 0$.

APPENDIX A: THE JUNCTION OF A COMOVING SOLUTION WITH A SCHWARZSCHILD EXTERIOR SOLUTION

The junction conditions are[6]

$$\Delta T^1_1 - \frac{dR}{dt}\,\Delta T^4_1 = 0\,, \tag{A1}$$

$$\Delta T^1_4 - \frac{dR}{dt}\,\Delta T^4_4 = 0\,, \tag{A2}$$

where $r = R(t)$ is the equation of motion of the junction surface. For a vacuum $T^u_v = 0$, while for a comoving perfect fluid $T^1_1 = -p$ and $T^4_4 = \rho$ so that (A1) and (A2) may be written in our case

$$-p = 0, \tag{A3}$$

$$\frac{dR}{dt}\,\rho = 0\,. \tag{A4}$$

Equations (A3) and (A4) can be satisfied in two ways:

(1) $p[R(t), t] \equiv 0$, $\rho[R(t), t] \equiv 0$, i.e., by finding a function $r = R(t)$ for which p and ρ become zero

APPENDIX B: DERIVATION OF SOLUTIONS OF THE EQUATION $T_2^2=0$

The relevant equation is[9]

$$e^{-\omega}(2\sigma''+\sigma'^2+2\mu''+\mu'^2-\mu'\omega'-\sigma'\omega'+\mu'\sigma')$$
$$+e^{-\sigma}(\dot\omega\dot\sigma+\dot\mu\dot\sigma-\dot\omega\dot\mu-2\ddot\omega-\dot\omega^2-2\ddot\mu+\dot\mu^2)=0\,. \quad \text{(B1)}$$

As a first step we look for the class of solutions in which each of $A=e^{\omega}$, $B=e^{\mu}$, and $C=e^{\sigma}$ is a product of a function of time by a function of r. In this case μ', ω', σ' are functions of r only while $\dot\mu$, $\dot\omega$, $\dot\sigma$ are functions of t only.

Writing with the evident notation $A=R_AT_A$, $B=R_BT_B$, and $C=R_CT_C$, Eq. (B1) becomes equivalent to two ordinary differential equations:

$$\frac{R_C}{R_A}(2\sigma''+\sigma'^2+2\mu''+\mu'^2-\mu'\omega'-\sigma'\omega'+\mu'\sigma')=a \quad \text{(B2)}$$

and

$$\frac{T_A}{T_C}(\dot\omega\dot\sigma+\dot\mu\dot\sigma-\dot\omega\dot\mu-2\ddot\omega-\dot\omega^2-2\ddot\mu-\dot\mu^2)=-a\,, \quad \text{(B3)}$$

where a is an arbitrary constant.

cording to a being equal to zero or not. The two equations (B2) and (B3) are still very complicated. However, the liberty of choosing the system of coordinates allows us to impose a condition on σ, ω, and μ. In order that the condition be a restriction on the choice of the coordinates only without restricting the generality of the solution, we must

in which h and R_B are arbitrary functions of r while g and T_B are arbitrary functions of t (u and v being arbitrary constants).

If we write $f=g(t)h(r)$, we may write the solution as follows:

$$A=\frac{uf'^2}{B}\exp\int\frac{B'^2f\,dr}{2B^2f'}\,,$$
$$B=RT\,, \quad \text{(B8)}$$
$$C=\frac{v\dot f^2}{B}\exp\int\frac{\dot B^2f\,dt}{2B^2\dot f}\,.$$

In (B8) B and f are restricted to be each one a product of a function of r by a function of t.

APPENDIX C: DERIVATION OF THE GENERALIZATION OF THE OS SOLUTION

The OS line element is given by

$$ds^2=-A\,dr^2-B(d\theta^2+\sin^2\theta\,d\psi^2)+dt^2\,, \quad \text{(C1)}$$

$$A=\frac{B'^2}{4B}=\left(\frac{d\sqrt{B}}{dr}\right)^2, \quad \text{(C2)}$$

Equation (C2) is characteristic of a flat 3-geometry and is a particular solution of the equation $T_4^1=0$ for the metric (C1). By adhering to Eq. (C2), we are restricting ourselves to solutions having a flat 3-geometry in comoving coordinates.

Taking into account that $T_4^1=0$, the equation $T^{\mu}_{1;\mu}=0$ takes the form

$$B=[\dot{\tau}(t)]^{-2/3}[F(r)\tau(t)+G(r)]^{4/3}. \qquad \text{(C9)}$$

APPENDIX D: CHARACTERIZING FRIEDMANN'S SOLUTION

We intend to show that the following statement is true: A spherically symmetric metric with cosmological time representing a comoving perfect fluid is a Friedmannian metric if $B=R(r)T(t)$; R and T being functions of their arguments and B being the metric coefficient occurring in

$$ds^2=-Adr^2-B(d\theta^2+\sin^2\theta d\psi^2)+dt^2. \qquad \text{(D1)}$$

The comoving condition as given by Tolman[10] is

$$A=(B'^2/4B)f(r) \qquad \text{(D2)}$$

and may be written with $B=R(r)T(t)$,

$$A=Tg(r). \qquad \text{(D3)}$$

The equation $T^1_1=T^2_2$ gives, in terms of T, R, and their derivatives,

$$-\frac{1}{RT}+\frac{1}{2Tg}\left(\frac{R'}{R}\right)^2=\frac{1}{2Tg}\frac{R''}{R}+\frac{1}{4Tg}\frac{g'R'}{gR}, \qquad \text{(D4)}$$

the solution of which is

$$g=\frac{R'^2}{4R+aR^2} \quad (a=\text{constant}). \qquad \text{(D5)}$$

Now, $R(r)$ may be chosen arbitrarily, if necessary, by a transformation $r=r(\bar{r})$, $t=\bar{t}$ which does not alter the comoving character of the line element; moreover, Eq. (D5) is covariant relative to such a transformation. Let us therefore take $R=r^2$; we obtain

$$g=\frac{1}{1+\frac{1}{4}ar^2}, \qquad \text{(D6)}$$

so that we can write

$$ds^2=-T\left[\frac{dr^2}{1+\frac{1}{4}ar^2}+r^2(d\theta^2+\sin^2\theta d\psi^2)\right]+dt^2, \qquad \text{(D7)}$$

which is one of the forms of Friedmann's solution.

We may also state: A spherically symmetric element with cosmological time representing a perfect fluid is a Friedmannian metric if $B=R_1T$ and $A=R_2T$ in an obvious notation.

It is easy to check that in this case $T^1_4=T^4_1=0$, and therefore, this second statement reduces to the preceding one.

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[2]C. Leibovitz Phys. Rev. 185, 1664 (1969).

[3]R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford Univ. Press, Oxford, England, 1934), p. 251.

[4]This function describes the total energy within coordinate r as a function of time. It has been introduced by Bardeen, Misner, and Sharp and has been used by Cahill and McVittie. We refer here more specifically to M. E. Cahill and G. C. McVittie, J. Math. Phys. 11, 1382 (1970).

[5]A. Einstein, Ann. Math. 40, 922 (1939).

[6]W. Israel, Proc. Roy. Soc. (London) 248A, 404 (1958).

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[8]The continuity of $P'(r)$ is necessary if we demand that A be continuous at a junction surface; however it has been shown (see Ref. 11) that A may be discontinuous in comoving coordinates provided we enforce the continuity of B'^2/A. In our case this last quantity is equal to $4B$, and its continuity is obtained by demanding the continuity of $P(r)$ only. The continuity of $P'(r)$ is therefore not necessary.

[9]L. Landau and L. Lifshitz, *Classical Theory of Fields*, (Addison-Wesley, Reading, Mass., 1951), p. 311.

[10]R. Tolman, private communication to J. Oppenheimer and H. Snyder (see Ref. 7).

[11]C. Leibovitz, Nuovo Cimento 60B, 254 (1969).