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A formal general solution of Einstein's equations in the static case containing an arbitrary function of r is obtained. A necessary and sufficient condition that the arbitrary function must satisfy in order that the solution be physically meaningful in the neighborhood of the center is established. A mapping from Newtonian solutions is indicated. The case of infinite pressure at the center is considered. New solutions are given as examples.

INTRODUCTION

MOST of the known exact solutions of Einstein's equations in the spherically symmetric static case have been found with *ad hoc* methods which may be described as follows: By manipulating Einstein's equations, a complicated differential equation is obtained connecting two unknown functions; the differential equation becomes simple, however, for particular forms of one of the functions.[1–5]

Although this method has no physical basis and may give unphysical results, it remains a valuable one in view of the scarcity of known exact solutions. It is felt, however, that there is a need for a more physical approach.

The present work is concerned with (a) properties of Einstein's equations and their solutions in the case of perfect fluids with spherical symmetry and (b) new exact solutions and their physical properties.

FORMAL GENERAL SOLUTION

We will use freedom in the choice of coordinates to take the following line element:

From $T_1{}^i{}_{;1}=0$ we obtain

$$-\tfrac{1}{2}\sigma'=p'/(p+\rho). \qquad (6)$$

We have to solve the equation $T_1{}^1=T_2{}^2$, which is a differential equation relating ω to σ. From the mathematical point of view as well as from the physical point of view, it is more convenient to consider σ as an arbitrary function of r and express ω in terms of σ; reverting to A and C, the equation $T_1{}^1=T_2{}^2$ reads

$$-\frac{A'}{A^2}\left(\frac{C'}{4C}+\frac{1}{2r}\right)+\frac{1}{A}\left(\frac{C''}{2C}-\frac{C'}{2rC}-\frac{C'^2}{4C}-\frac{1}{r^2}\right)+\frac{1}{r^2}=0, \qquad (7)$$

which is a linear differential equation in A^{-1}, the solution of which is

$$A=e^{\omega}=r^{-2}e^{\sigma}(2+\sigma'r)^2\exp\left(-4\int\frac{\sigma'dr}{2+r\sigma'}\right)\Bigg/$$
$$-4\int\left[r^{-3}e^{\sigma}(2+r\sigma')\exp\left(-4\int\frac{\sigma'dr}{2+r\sigma'}\right)\right]dr+\text{const.} \qquad (8)$$

If we choose the a's and the b's so that every $4c_i$ is a whole number, all the integrations can be performed, and we obtain solutions expressed in terms of known functions.

However, we are interested in finding conditions for the choice of σ so that the resulting solutions will be physically meaningful. We shall do this in three different ways and shall establish conditions on σ so as to obtain (a) physically meaningful solutions with finite pressure at the center, (b) a mapping of Newtonian solutions, and (c) solutions with infinite pressure at the center that may be used in the problem of collapse.

Conditions for Physical Solutions with Finite Pressure at Center

From the three equations (3), (5), and (6) we can eliminate A and C to obtain

$$\begin{aligned}&8\pi r^3(-2p'^2\rho+4pp'^2+2pp'\rho'-2p\rho p''-2p^2p'')\\&+3(8\pi)r^2p'(\rho+p)^2+r[6p'^2-2p''(\rho+p)+2p'\rho'\\&-8\pi(\rho+p)^2(\rho+3p)]-4p'(\rho+p)=0.\quad(12)\end{aligned}$$

It is known[7] that in the static case and for the metric (1) the pressure is a decreasing function of r; therefore, the maximum of the pressure is at the origin; if an infinity of the pressure is to be avoided it therefore is enough to have finite pressure at the origin. We shall now prove the following results:

Result I. If the origin is a regular point for ρ and p, then at the origin we must have $p_0'=0$.

If the origin is a regular point for ρ and p, they can be expanded in a Taylor series. Introducing these series into Eq. (12) and setting $r=0$, we obtain

$$-4p_0'(\rho_0+p_0)=0, \quad (13)$$

so that we must have $p_0'=0$.

Result II. If the origin is a regular point for ρ and p as functions of r, then ρ and p must be even functions of r.

The proof of this result is given in the Appendix.

Result III. If the origin is a regular point for $\rho(r)$ and if A^{-1} behaves there as ar^n with $n\neq-1$, then we must have at the origin $A=e^{\omega}=1$.

From (5) we may write

$$8\pi r^2\rho+(A^{-1}-1)-A'r/A^2=0. \quad (14)$$

Assuming that A^{-1} behaves like ar^n for small r, we have at the origin for finite ρ

$$(n+1)ar^n-1=0,$$

so that $n=0$ and $a=1$, that is to say, $A(0)=1$.

Result IV. If the origin is a regular point for ρ, then A is an even function of r.

[7] B. Harrison, K. Thorne, M. Wakano, and J. Wheeler, *Gravitational Theory and Gravitational Collapse* (The University of Chicago Press, Chicago, 1965), p. 21.

The result is evident upon inspecting Eq. (5) and taking result II into account.

Result V. If the origin is a regular point for ρ and p, we must have

$$p_0''=-(\tfrac{4}{3}\pi)(\rho_0+p_0)(\rho_0+3p_0). \quad (15)$$

Let us introduce in (12) the Taylor expansion of ρ and p in terms of even powers of r (result II), and let us set equal to zero the coefficient of r in the resulting equation; we obtain

$$-2p_0''-(\rho_0+p_0)8\pi(\rho_0+3p_0)-4p_0''=0,$$

which leads to the result.

What is remarkable about this result is that it is independent of the equation of state.

Result VI. If ρ and p are regular at the origin and if A^{-1} does not behave there as r^{-1}, then σ and therefore $C=e^{\sigma}$ admit a Taylor expansion around the origin in even powers of r.

We have from (3)

$$\sigma'=A(8\pi pr+1/r)-1/r. \quad (16)$$

Introducing the Taylor expansion of p and A in even powers of r (results II and V) and taking III into account, we obtain for σ' a Taylor expansion in odd powers of r, the integral of which is a Taylor series in even powers of r.

Now, let us write the equation of state in the form

$$\rho=\rho_0+a(p-p_0)+b(p-p_0)^2+\cdots, \quad (17)$$

and expand p, ρ, A, and σ around the origin:

$$p=p_0-\tfrac{1}{12}(\rho_0+p_0)8\pi(\rho_0+3p_0)r^2+dr^4+\cdots, \quad (18)$$

$$A=1+ur^2+vr^4+wr^6+\cdots, \quad (19)$$

$$\rho=\rho_0+fr^2+gr^4+hr^6+\cdots, \quad (20)$$

$$\sigma=\sigma_0+jr^2+kr^4+\cdots. \quad (21)$$

Introducing these expressions into (6), (3), and (5), we obtain

$$j=\tfrac{4}{3}\pi(\rho_0+3p_0), \quad (22)$$

$$u=8\pi\rho_0/3, \quad (23)$$

$$a=\frac{5}{3}\frac{\rho_0-3p_0}{\rho_0+p_0}-\frac{240k}{(8\pi)^2(\rho_0+p_0)(\rho_0+3p_0)}. \quad (24)$$

We may therefore write for σ

$$\sigma=\sigma_0+\tfrac{4}{3}\pi(\rho_0+3p_0)r^2+(8\pi)^2\times\frac{(\rho_0+p_0)(\rho_0+3p_0)}{240}\left(\frac{5}{3}\frac{\rho_0-3p_0}{\rho_0+p_0}-a\right)r^4+O(r^6). \quad (25)$$

We therefore have the following indications on σ:

(1) σ is an even function of r.

(2) The constant σ_0 is of course arbitrary and is changed by a rescaling of time.

(3) The coefficient of r^2 in (21) must be positive ($j>0$).

(4) The choice of the coefficient j of r^2 determines the value of ρ_0+3p_0. The knowledge of σ cannot determine ρ_0 and p_0 independently. Concerning this last point, we have from (8) [and taking (19) and (23) into consideration] that

$$A=\frac{r^2X(r)}{r^2Y(r)+\text{const}\times r^2}=1+\frac{8\pi\rho_0}{3}r^2+\cdots, \qquad (26)$$

where $r^2X(r)$ and $r^2Y(r)$ tend to a constant different from 0 for $r=0$. The constant in A is therefore given by

$$-(8\pi\rho_0/3)[r^2X(r)]_{r=0}. \qquad (27)$$

It is therefore clear that the value of the constant in A affects the value of ρ_0. In short, the knowledge of σ

[illegible]

determines the solution up to an arbitrary constant in A; this allows for an arbitrary determination of ρ_0; the solution is therefore completely determined by giving σ and ρ_0.

However, once σ (and therefore j) is given, ρ_0 must be restricted in its values for physical reasons; we have from (22)

$$\rho_0=(3j/4\pi)-3p_0. \qquad (28)$$

We will therefore impose on ρ_0 the inequalities

$$3j/16\pi\leqslant\rho_0\leqslant 3j/4\pi, \qquad (29)$$

which correspond to the physical inequalities

$$\rho_0\geqslant p_0\geqslant 0. \qquad (30)$$

We shall impose also the condition $(d\rho/dp)_0\geqslant 1$, i.e. [from (17) and (24)],

$$\frac{5}{3}\left(\frac{\rho_0-3p_0}{\rho_0+p_0}\right)-\frac{240k}{(8\pi)^2(\rho_0+p_0)(\rho_0+3p_0)}\geqslant 1. \qquad (31)$$

[illegible]

around the origin we have $d\rho/dp\geqslant 1$ and $\rho_0\geqslant p_0\geqslant 0$, is that σ admit a Taylor expansion

$$\sigma=\sigma_0+jr^2+kr^4+\cdots$$

in even powers of r, and that j and k satisfy the inequalities

$$j>0 \quad\text{and}\quad k\leqslant\tfrac{1}{10}j^2 \qquad (34)$$

—or equivalently, that C admit a Taylor expansion

$$C=m(1+nr^2+qr^4+\cdots) \qquad (35)$$

in even powers of r with

$$n>0 \quad\text{and}\quad q\leqslant\tfrac{3}{5}n^2. \qquad (36)$$

Let us apply result VII on the different exact solutions mentioned in Tolman's article.[1]

Solution I is that of Einstein's universe. We have [illegible]

This solution does not, therefore, comply with the conditions (34) or (36) of result VII (hereafter referred to simply as "our conditions"). It is known that for this solution $\rho+3p=0$, unless we use the cosmological constant.

Solution II is that of Schwarzschild and de Sitter; we have in this case

$$e^{\sigma}=\text{const}(1-2M/r-r^2/R^2). \qquad (37)$$

If $M\neq 0$, the solution does not respect our conditions which state that e^{σ} is to have a Taylor expansion around the origin. In fact, in such a case A^{-1} behaves like r^{-1} at the origin. We have excluded this case in establishing result III. (We have therefore excluded the case of a point mass at the origin.)

The case $M=0$ gives $n=-1/R^2$ and $q=0$, which again contradict our conditions. In fact, we have in this case de Sitter's universe, for which $\rho+p=0$.

Solution III is the Schwarzschild interior solution. We have in this case

[illegible]

The most general expression for e^{ω} may be calculated from (8), and is found to be

$$e^{\omega}=\frac{[x(A-Bx)+B(1-x^2)]^2[(x-x_1)^{x_1}/(x-x_2)^{x_2}]^{2/(x_2-x_1)}}{[x(A-Bx)+B(1-x^2)]^2[(x-x_1)^{x_1}/(x-x_2)^{x_2}]^{2/(x_2-x_1)}x^2+ar^2}, \tag{40}$$

in which

$$x=\left(1-\frac{r^2}{R^2}\right)^{1/2},\quad x_1=\frac{A-(A^2+8B^2)^{1/2}}{4B},\quad x_2=\frac{A+(A^2+8B^2)^{1/2}}{4B}, \tag{41}$$

while a is the additional arbitrary constant. The Schwarzschild interior solution corresponds to the particular value $a=0$.

If we give to A/B the value 2, for instance, we shall have satisfied the imposed inequalities. We can then choose the arbitrary constant a so as to have $\rho_0>p_0>0$.

Solution IV is the most interesting of Tolman's solutions. We have in this case

Tolman, who imposed the inequality $\rho_0/p_0\geqslant 3$ for which we must have $R^2>0$.

Solution V: We have here $e^{\sigma}=r^{2n}$; σ cannot be expanded into a Taylor series around the origin so that it does not respect our conditions. In fact, the solution has infinite pressure and density at the origin.

Solution VI:

$$e^{\sigma}=(Ar^{1-n}-Br^{1+n})^2,\quad e^{\omega}=2-n^2, \tag{48}$$

$$8\pi\rho=(1-n^2)/r^2(2-n^2), \tag{49}$$

$$8\pi p=[(1-n)^2A-(1+n)^2Br^{2n}]/r^2(2-n^2)(A-Br^{2n}). \tag{50}$$

As given by Tolman, this solution is singular except for $n^2=1$, for which we have $\rho=0$ and $p\neq 0$, an unacceptable situation.

our criterion. In fact, the expression for [illegible] as given [illegible] We could have taken for [illegible] the expression [illegible]

Taking $C=r^{4/(1+k)}$, we find from (8)

$$A=\frac{(k^2+6k+1)/(k+1)^2}{1-ur^{(2k^2+12k+2)/(k^2+4k+3)}}, \qquad (67)$$

$$8\pi p=-\frac{5+k}{1+k}\,\frac{(k+1)^2u}{k^2+6k+1}r^{(4k-4)/(k^2+4k+3)}+\frac{4}{k^2+6k+1}r^{-2}, \qquad (68)$$

$$8\pi\rho=\frac{3k^2+16k+5}{k^2+4k+3}\,\frac{(k+1)^2u}{k^2+6k+1}r^{(4k-4)/(k^2+4k+3)}$$

4k

while the total mass is given by

$$m=\tfrac{1}{3}(3u)^{-1/2}. \qquad (76)$$

It is clear that with u small enough, the total mass can be as great as desired. (The relevance of this solution to the problem of gravitational collapse will be the subject of another paper.)

DISCUSSION

We have adopted as the criterion for physicality the condition that pressure and density be finite at the for a solution of

$$f(y,y',y'',x)=0, \tag{A3}$$

then for this solution we have

$$y(-x)=y(x). \tag{A4}$$

The proof consists in the remark that $x\to y(-x)$ also satisfies (A3) and has vanishing derivative at $x=0$, hence $y(-x)=y(x)$ because of uniqueness.

From this theorem we deduce the truth of result II. A look at Eq. (12) supplemented by $\rho=f(p)$ shows that the left side of the Eq. satisfies (A1); now, result I states that $p'(0)=0$ and therefore the solution $p(r)$ of (12) is an even function of r.

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Classical Relativistic Rotator as a Basis for the Elementary Particles*

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A classical Lorentz-covariant generalization of the nonrelativistic theory of a free, stationary, symmetric top is developed. The resulting relativistic theory predicts a physical mass which is a monotonically increasing function of spin asymptotically approaching a linear relation in the limit of large spin. The theory is free of spacelike solutions.

I. INTRODUCTION

THE notion that the elementary-particle resonances may be excited rotational states is not new. It has led to the investigation of the rotational levels of composite systems and to the study of relativistic wave equations based on various rotator models.[1] Perhaps the most detailed study of the applicability of rotational states to the elementary particles is due to Corben.[2] His analysis is based on the model of a symmetric top. It is in the spirit of Corben's approach, that a properly formulated quantum theory of a relativistic rotator is founded on a properly formulated classical theory of that same rotator, that we undertake the present analysis. Some of the introductory material has appeared in the literature; it is reiterated, in Secs. I and II, for the sake of coherence. The rest of the analysis and the emerging rotator theory differ in content from previous formulations.

We develop a classical Lorentz covariant generalization of the free, nonrelativistic, symmetric top and discuss those features of the relativistic theory which indicate its relevance to the elementary particles. We focus especially on the two important features: (a) the predicted relation between the physical mass and spin of the rotator, and (b) the question of spacelike solutions.[3] These two crucial aspects prove to be directly related in our formulation, for the condition which ensures that the physical mass increase monotonically with spin also rules out the possibility of spacelike four-momenta.

Our present purpose is only to indicate the relevance of the model of the symmetric top to a discussion of the elementary particles. Therefore, in the nonrelativistic theory, we make the relatively simple choice of collinear spin angular momentum $\mathbf{S}$ and angular velocity $\boldsymbol{\omega}$. Thus the rotational kinetic energy in the nonrelativistic theory is

$$T=\tfrac{1}{2}\mathbf{S}\cdot\boldsymbol{\omega} \;=\; S^2/2I, \tag{1}$$

where I is the moment of inertia about the axis of rotation.[4] The energy-spin relation (1) forms the basis for a highly successful quantum theory of the rotational levels of symmetric molecules and heavy symmetric nuclei.[5] This quantum theory follows almost trivially by merely replacing the classical spin variable $\mathbf{S}$ by $\hbar\mathbf{J}$, where $\mathbf{J}$ is the spin operator in units of $\hbar$. Equation (1) then gives the energy eigenvalues for states of well-defined angular momentum with the continuous classical variable S^2 replaced by the discrete values $\hbar^2 j(j+1)$.

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[1] D. Bohm, P. Hillion, T. Takabayasi, and J. P. Vigier, Progr. Theoret. Phys. (Kyoto) **23**, 496 (1960); T. Takabayasi, *ibid.* **23**, 915 (1960); **36**, 660 (1966); E. J. Sternglass, Phys. Rev. **123**, 391 (1961); Nuovo Cimento **35**, 227 (1965); H. C. Corben, Proc. Natl. Acad. Sci. (U.S.) **48**, 1559 (1962); **48**, 1746 (1962); J. Math. Phys. **5**, 1664 (1964); Nuovo Cimento **47**, 486 (1967); M. Gell-Mann, Phys. Today **17**, 23 (1964); A. O. Barut, *High Energy Physics and Elementary Particles* (International Atomic Energy Agency, Vienna, 1965), p. 679; K. Rafanelli, Phys. Rev. **175**, 1947 (1968).

[2] H. C. Corben, *Classical and Quantum Theories of Spinning Particles* (Holden-Day Inc., San Francisco, 1968).

[3] E. Abers, I. T. Grodsky, and R. Norton, Phys. Rev. **159**, 1222 (1967); I. T. Grodsky and R. F. Streater, Phys. Rev. Letters **20**, 695 (1968); S. J. Chang and L. O'Raifeartaigh, Phys. Rev. **170**, 1316 (1968).

[4] H. Goldstein, *Classical Mechanics* (Addison-Wesley Publishing Co., Reading, Mass., 1959), Chap. 5.

[5] G. Herzberg, *Infrared and Raman Spectra of Polyatomic Molecules* (D. Van Nostrand, Inc., Princeton, N. J., 1945); K. Kotajima and D. Vinciguerra, Phys. Rev. Letters **8**, 68 (1964); L. D. Landau and E. M. Lifshitz, *Quantum Mechanics, Non-Relativistic Theory* (Addison-Wesley Publishing Co., Reading, Mass., 1958), p. 373.