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Maximum Efficiency of Energy Release in Spherical Collapse*

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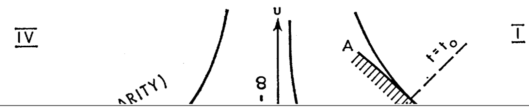
The question of the maximum amount of energy which can be radiated by a collapsing spherical star is reexamined. On Newtonian theory, gravitational energy is negative and unbounded below, so that unlimited amounts of energy can be released. It is shown that, in the relativistic collapse of a star with non-negative energy density, self-closure always takes place before the star can release 100% of its initially positive mass energy. Moreover, under physically reasonable restrictions on the pressure, the 100% upper limit can be approached only if the star happens to pass through a very special and improbable momentarily static configuration first considered by Zel'dovich. It is concluded that for normal spherical collapse the efficiency of energy release must be low.

I. INTRODUCTION

EARLY attempts to explain the prodigious energy

total gravitational mass is only $\Delta m = \Delta M (1-2m/r)^{1/2}$, the reduction arising from the loss of potential energy. It follows that the gravitational mass of an arbitrarily since the positive-energy condition ($T_\nu{}^\mu u_\mu u^\nu \geq 0$ for all timelike vectors u^μ) implies $T_4{}^4 \leq 0$. We write $r=R(t)$ for the boundary with the exterior vacuum, so that $m[R(t), t] \equiv m$ is the externally observed gravitational



an immediate consequence of $g^{\alpha\beta}u_\alpha u_\beta=-1$. By choosing the expression (5) (more accurately, its mass average over the star) sufficiently small, and *only* by so doing, we can reduce the first integral to arbitrarily small values for any given number of baryons A and any given equation of state.

The necessary and sufficient condition that $m\approx 0$ for fixed A is therefore essentially that the star pass through a momentarily static configuration with mass distribution given by the Zel'dovich condition[3] $r=2m(r,t)+\epsilon(r,t)$ $(\epsilon\to 0+)$, i.e., $\mu\approx 1/8\pi r^2$. This distribution could be truncated internally so as to form a hollow shell[8]; if it is continued inwards to the center, $\epsilon(r,t)$ must be adjusted carefully to avoid a singularity there.[3]

IV. NATURE OF "UNIFORM CONFIGURATIONS"

An apparent counterexample to the result just proven is to be found in Ref. 9, where a sequence of momentarily static configurations is exhibited whose mass tends to zero for arbitrarily large A, and which have *uniform* density. However, as we shall now show, these models have unusual properties which make them inapplicable in the present context.

The momentarily static configurations constructed in Ref. 9 are characterized by $\mu(r,t_0)=\mu_0=\text{const}$, so that $m(r,t_0)=\frac{4}{3}\pi\mu_0 r^3$. Introducing a new radial coordinate χ, defined by

$$r=a\sin\chi$$

into (1), where $a\equiv[(8/3)\pi\mu_0]^{-1/2}$, we find

$$(ds^2)_{t=t_0}=a^2(d\chi^2+\sin^2\chi\, d\Omega^2)-e^{2\psi}\cos^2\chi\, dt^2, \qquad (6)$$

showing that $a\chi$ measures radial proper distance from the center. The boundary of the configuration is given by

$$\chi=\chi_0, \quad r=R(t_0)=a\sin\chi_0, \quad m=\tfrac{1}{2}a\sin^3\chi_0. \qquad (7)$$

The authors of Ref. 9 now argue that if χ_0 is allowed to increase towards π, the baryon number

$$A=4\pi a^3\int_0^{\chi_0} n\sin^2\chi\, d\chi$$

increases towards a finite limit, whereas (7) shows that both the radius and the gravitational mass sink to zero.

To understand the origin of this result, we must examine more closely the nature of the configurations with $\chi_0>\frac{1}{2}\pi$. These models can be interpreted in two possible ways.

According to the first interpretation, the star is (at least initially) an accessible component of "our" universe with boundary represented by a curve, such as AB, in quadrant I of Fig. 1. However, if such a configuration (with $\chi_0>\frac{1}{2}\pi$) were truly a distribution of uniform density μ_0, as we might at first suppose, then we run into the following difficulty: Adding a spherical shell of uniform density μ_0 should lead to a new "uniform" configuration whose radius R is both *larger* than the original (since it extends further into the exterior Schwarzschild space where r is monotonically increasing) and also *smaller* (since it corresponds to a larger A and larger χ_0).

At the root of this difficulty is clearly the fact that r changes abruptly from decreasing outwards to increasing outwards at the boundary of a configuration with $\chi_0>\frac{1}{2}\pi$, and this indicates the presence of a shell of mass.

We adopt θ, ϕ, and proper time τ as intrinsic coordinates θ^a of the boundary, so that the intrinsic three-metric is

$$g_{ab}d\theta^a d\theta^b=R^2d\Omega^2-d\tau^2.$$

The surface energy three-tensor S_{ab} of the shell is given in terms of the jump $\gamma_{ab}=K_{ab}^{+}-K_{ab}^{-}$ of the extrinsic curvature by[10]

$$-8\pi S_{ab}=\gamma_{ab}-g_{ab}g^{cd}\gamma_{cd}.$$

We thus derive the surface density

$$S_{\tau\tau}=-(1/4\pi R^2)\gamma_{\theta\theta}.$$

A straightforward calculation yields for the extrinsic curvatures of the imbeddings in the interior space (6) and the exterior Schwarzschild space at the moment of the time symmetry $t=t_0$

$$K_{\theta\theta}^{-}=a\sin\chi_0\cos\chi_0,$$
$$K_{\theta\theta}^{+}=R(1-2m/R)^{1/2}=a\sin\chi_0|\cos\chi_0|.$$

Substitution into (8) shows that $S_{\tau\tau}<0$ if $\chi_0>\frac{1}{2}\pi$.

We conclude that the "uniform" configurations with $\chi_0>\frac{1}{2}\pi$, if regarded as part of "our" universe, are actually encased in a layer of negative mass.

The second interpretation of the $\chi_0>\frac{1}{2}\pi$ models avoids this particular difficulty. The boundary is considered to be a curve, such as CD, in quadrant III of Fig. 1. This makes the gradient of r continuous across the boundary. However, the exterior vacuum region now includes two singular curves $r=0$ and both past and future event horizons $r=2m$, $t=\pm\infty$. To an external observer such an object would appear simply as a "black hole" which has existed in that form for all time, and it is specifically excluded from the considerations of Sec. III by our assumption of nonsingular initial conditions.

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[8] For *thin* shells in equilibrium the fact that $m\to 0$ in the ultrarelativistic limit has been noted and discussed in some detail by J. E. Chase [Nuovo Cimento (to be published)].

[9] B. K. Harrison, K. S. Thorne, M. Wakano, and J. A. Wheeler, *Gravitation Theory and Gravitational Collapse* (University of Chicago Press, Chicago, 1965), Chap. 8.

[10] W. Israel, Nuovo Cimento **44B**, 1 (1966); **48B**, 463 (1967).