VARIABLE G WITHIN A MODIFIED THEORY OF GENERAL RELATIVITY

(Letter to the Editor)

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Abstract. The case is made for modifying the equations of general relativity so as to permit a time-variable gravitational 'constant'.

1. Introduction

The case has been made for a time-dependent gravitational constant in order to explain the order of magnitude of some numbers which characterize our Universe (Wesson, 1980). However, this hypothesis would have disturbing consequences including the following:

(a) A time-dependent gravitational constant would imply a time variation of the laws of nature. Nature would no longer be 'conserved' under a time translation.

(b) Einstein's tensor G^a_b has been constructed to be divergenceless so as to ensure the divergenceless of the matter tensor T^a_b. This property is required for conservation laws considerations. However, with a variable gravitational 'constant', kT^a_b would still be divergenceless but T^a_b would then have a divergence different from zero.

(c) Einstein's equations would cease to be covariant.

(d) [illegible]

2. Justification for Modifying General Relativity

In our view, there may be more compelling reasons than the LNH for modifying Einstein's equations of general relativity. We consider two such reasons having to do with centrifugal force and numerical equalities.

2.1. General relativity and centrifugal forces

The Einstein equations being covariant, it follows that whether the Universe turns about an observer or whether an observer turns about himself, the centrifugal forces would be the same in both cases. This is not as straightforward a statement as it seems to be. If the two situations are related by a coordinate transformation, then we are facing a single physical situation expressed in two different frames of reference.

The problem becomes much more interesting when it addresses the relation between the following two different physical situations. In the first case, the observer rotates relative to the masses which are fixed relative to asymptotically inertial frames. In the second case, the masses rotate relative to the observer who is fixed relative to asymptotically inertial frames.

The first case is that which generates the well-known centrifugal forces. The second case was analysed by Thirring and Lense (1918). They found that a spherical shell rotating about an observer in the background of asymptotically fixed inertial frames, would produce a centrifugal force with the correct dependence on distance and angular velocity, but with a strength which depends on M/R, M being the mass of the spherical shell and R its radius.

In order for the centrifugal forces to be identical in the two cases, the following relation must hold (Møller, 1972):

$$k = 4\pi R/Mc^2 . \tag{2.1}$$

This relation holds in Einstein's static universe but does not hold classically in an expanding universe since it equates a constant k to the variable $4\pi R/Mc^2$.

2.2. Unlikely coincidence

At the time of the Big Bang, the right-hand side of Equation (2.1) is expected to have been close to zero, and thereafter has been growing with time as the Universe expands. At some time in the history of a big bang Universe, Equation (2.1) will be momentarily satisfied, and it so happens that time is...NOW.

Equation (2.1) can be rewritten as

$$k = 2H^2/(\pi\rho c^4) \tag{2.2}$$

in which H is the Hubble constant; ρ, the mean density of matter; and c, the speed of light. We moved from Equation (2.1) to Equation (2.2) by replacing M by $2\rho\pi^2R^3$ (Tolman, 1934) and R by c/H (the value for which the speed of recession equals that of light).

If we take for ρ the value 1.4×10^{-30} c.g.s. units, for h the value 2.5×10^{-18}

(75 km s^{-1} Mpc^{-1}), for c the value 3.0×10^{10}; the right-hand side of Equation (22) gives us 3.4×10^{-48} which, in view of the uncertainties in ρ and H, is in surprisingly good agreement with the measured value of 2.076×10^{-48} (Lang, 1980). Summarizing, we see that there are good reasons to expect Equation (2.1) to be true (reciprocity of rotational effects) and, it does happen to hold now. It is, therefore, natural to suspect that Equation (2.1) is indeed a relation holding for *all* times and not, by pure chance, just at our time.

In this letter we show that it is possible to modify Einstein's equations so that Equation (2.1) is not an ephemeral relation. Our efforts tend also to meet the objections (a)–(d) mentioned before. We note that the values of ρ and H remain uncertain and [redacted] fraction of the total (Maddox, 1984) and that the appropriate numerical value may be 10, or more, times larger than that given above. The value of H could be as low as 50 km s^{-1} Mpc^{-1} (Sandage and Tammann, 1982, 1984) or as high as 115 km s^{-1} Mpc^{-1} (de Vaucouleurs and Peters, 1986). These uncertainties do not affect our argument; the equality identified above still holds to order of magnitude.

the mass of a proton and its Compton wavelength, we could use them as units and find out that there was a real physical scaling up or down. But a proton is not an integral part of general relativity and, in its absence (or in the absence of any other fixed reference), one should expect that if all lengths, for instance including the measuring ones, were replaced by lengths twice as long, this would have undetectable physical effects. However, comparing Equations (3.1) and (3.3) shows that such is not the case. To avoid this situation (supposing that it should be avoided) we look for untis of length and density of matter *determined by the distribution of masses*. Expressed in those units, Einstein's equations would not need a dimensional constant k to relate geometry to matter since both would be non-dimensional.

Let us therefore, express all lengths in terms of the radius R of the Universe, and all matter densities in terms of the mean density of matter in the Universe. Einstein's equations become (taking the non-dimensional constant equal to unity)

$$G_b^{(3)a} = -T_b^{(3)a}, \tag{3.4}$$

with

$$G_b^{(3)a} = R^2 G_b^{(0)a} \quad \text{and} \quad T_b^{(3)a} = T_b^{(0)a}/\rho_0 c^2 . \tag{3.5}$$

Replacing $G^{(3)}$ and $T^{(3)}$ by their expressions in (3.5) we obtain

$$G_b^{(0)a} = -1(R^2 \rho c^2) T_b^{(0)a} . \tag{3.6}$$

We are getting back the original equations of general relativity except for k not being constant and being given by an expression essentially consistent with Equation (2.1) (replace ρ by M/R^3). Equations (3.4) and (3.6) are less general than Einstein's equations. They presuppose a spherical, finite universe and describe the evolution of the gravitational field within such a universe. We also note that in Equation (3.6) it is $R^2 \rho T^{(0)}$ which is divergenceless and not $T^{(0)}$.

4. An Alternative Solution

We are considering an alternative to Einstein's equations. It is based on covariant equations, accepts Einstein's equations as a first approximation, and leads to a variable gravitational constant. It is expressed by the equations

$$[g^{uv}(G_a^c)^{-1} T_c^b]_{;uv} = T_a^b . \tag{4.1}$$

The left-hand side of Equation (2.1) is the covariant Laplacian of

$$(G_a^c)^{-1} T_c^b .$$

In a quasi-flat space, Equation (4.1) can be approximated by

$$\Box (G_a^c)^{-1} T_c^b = T_a^b . \tag{4.2}$$

If T is zero at infinite distance or if we consider the waveless solution of Equation (4.1), the solution of Equation (4.2) is then given by

$$(G_a^c)^{-1} T_c^b = -(4\pi)^{-1} \int (r)^{-1} T_a^b \, \mathrm{d}v \,. \tag{4.3}$$

By contracted multiplication with G_j^a we obtain

$$T_j^b = -(4\pi)^{-1} \int r^{-1} T \, \mathrm{d}v G_j^b \,, \tag{4.4}$$

in which r is the time-retarded distance.

Equation (4.4) would be identical to Einstein's equations if

$$k^{-1} = (4\pi)^{-1} \int r^{-1} T \, \mathrm{d}v \,. \tag{4.5}$$

In a homogeneous universe, k, as defined by Equation (4.5), would vary little in space while varying with the expansion of the Universe in time, as we show below.

The integral

$$\int r^{-1} T \, \mathrm{d}v$$

is similar to a gravitational potential and, in a homogeneous universe, is of the order of $c^2 M/R$ in which M is the mass of the Universe and R its radius. If we replace the integral by this expression, Equation (4.5) becomes

$$k^{-1} = (4\pi)^{-1} M/r \quad \text{or} \quad k = 4\pi R/(Mc)^2 \,. \tag{4.6}$$

It is worth noting that k does not figure at all in Equation (4.1). As defined in Equation (4.6), its value can be predicted by the theory.

5. Singularities

The value of k as defined by Equation (3.4) varies very little from point to point. This is due to the fact that the contributions to its value as given in Equation (4.5) by any close-by object, are negligible compared to the homogeneous contribution of the whole Universe. Near the surface/edges of normal planets, stars, and galaxies, the universal value of k is decreased by less than one part in 10^6, or so. However, this ceases to be true close to a singularity. If we define by k_0 the contribution of all the Universe (except the singularity) and if the point-singularity is that of a mass m and if r indicates the distance between a point and the singularity, then according to Equation (3.3), we should have

$$k^{-1} = k_0^{-1} + (4\pi)^{-1} (mc^2/r) \,. \tag{5.1}$$

The Schwarzschild expression $(1 - 2mr^{-1})$ is, in our units,

$$1 - 2mkc^2 (8\pi r)^{-1} \quad \text{or} \quad 1 - mkc^2 (4\pi r)^{-1} \,. \tag{5.2}$$

If we replace k in (5.2) by its value in Equation (3.6) we obtain for the Schwarzschild term

$$1 - mc^2k_0/(4\pi r + mc^2k_0)\,. \tag{5.3}$$

As long as r is different from zero, this expression remains positive. It becomes zero only at the singularity itself. The Schwarzschild singularity has disappeared. We must point out that the preceding treatment for the singularity is invalid since, at a short distance from it, the assumptions that allowed us to derive Equations (4.4) from Equations (4.1) are no longer valid. The treatment, though lacking in rigor, does indicate that singular masses might not necessarily produce a singular surface way from the singularity.

Equation (4.1) is not the only covariant equation leading to a variable gravitational 'constant'. The following equation leads to a variable gravitational tensor 'constant'

$$[g^{uv}G_a^{-1c}T_c^b]_{;\,uv} = T_a^b\,. \tag{5.4}$$

Equation (4.5) then becomes

$$k_b^{-1a} = (4\pi)^{-1}\int r^{-1}T_b^a\,\mathrm{d}v \tag{5.5}$$

so that, instead of Einstein's equations, we obtain in our approximation

$$k_b^{-1a}G_c^b = -T_c^a\,. \tag{5.6}$$

In the spherically-symmetric case expressed in spherical coordinates, the value of k at any time would be different for the expression of the radial pressure as compared to the expression for the lateral pressure or the density. This, unless it so happens that

$$T_c^a = ctte_c^a\,, \tag{5.7}$$

which would also means that $\rho + p = 0$.

It is to be noted that precisely such a relation is given by the cosmological constant and was suggested by Møller (1972) to eliminate 'side-effects' in Thirring's computations.

6. The Nature and Consequences of Variable G

According to Equation (4.6) the gravitational constant *increases* with time. Dirac expected it to decrease. The variability of the 'constant' would have dramatic effects on cosmology. According to Equation (3.5), the gravitational constant was much lower in the early stages of the Universe. These stages may, therefore, have lasted for a shorter time than in the Einsteinian case, since the expansion was resisted by smaller attractive forces. Likewise, the gravitational forces are increasing with time as compared to the Einsteinian predicted values. This may increase the likelihood of a closed or oscillating universe.

It is, however, in the field of gravitational light deflection that the effect of the variation would be easiest to observe. Close to a singularity the deflection would be smaller than predicted by Einstein's equations. Since the pioneering work of Eddington (1919) and others, many attempts have been made to measure the gravitational light deflection. The classical approach involved measuring the deflection of visible radiation from background stars near the limb of the eclipsed Sun. The most precise results have been obtained during recent decades at radio wavelengths while monitoring man-made satellites in heliocentric orbits and observing galactic and extragalactic radio sources in the solar background. This work has been frequently discussed and reviewed in the literature, and the most reliable results appear to be in very close agreement with the predictions of classical general relativity (Fomalont and Sramek, 1975).

The angular deflection of electromagnetic radiation is proportional to the gravitational constant (Lang, 1980; Misner *et al.*, 1973) and, therefore, to k. In principle then, a time-dependence of the gravitational 'constant' will reveal itself in, for example, the recently detected gravitational lenses which involve quasars and distant – in time as well as space – galaxies. In practice, the difficulties involved in disentangling this effect from other factors which must be considered in interpreting the observations are currently insurmountable.

7. Conclusions

A relation required for reciprocity of rotational effects and which is reasonably satisfied today, may represent a fundamental property of nature. Using this relation, it has been shown that it is possible to formulate solutions to Einstein's equations that overcome some of the objections raised against a variable gravitational constant. The proper theoretical way, if such exists, may still remain to be found. While we have demonstrated that variable G can be accommodated within a modified theory of general relativity – and that was the purpose of this exercise – it remains true that there is very little convincing *observational* evidence for or against a time-variable gravitational constant.

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